The Hierarchy of Univalent Universes and a Type of Strictly Homotopy Level n

Nicolai Kraus

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Abstract

In Martin-Löf Type Theory with a hierarchy $U_0, U_1, U_2, \ldots$ of univalent universes, it is well-known that the first universe U_0 is not a 0-type (an h-set). For the fact that U_1 is not an 1-type, several people have found various solutions. In this note, we solve the general case and show that for all $n \in \mathbb{N}$, the universe U_n is not an n-type. At the same time, we give an answer to the (very related) problem of constructing an $n + 1$-type that is not an n-type, which was listed as an open problem of the Univalent Foundations Program in Princeton.

We have formalized the proof in Agda.[1]

Notation

We are working in "standard" MLTT (with Σ-, Π- and Identity-types). Further, we assume that there is a hierarchy of univalent universes, $U_0, U_1, U_2, \ldots$, where U_0 is the lowest universe (containing the natural numbers - we have no universe of propositions in this hierarchy).

If we can prove a lemma for all universes at the same time, we keep the index implicit and just write U. The corresponding statement is to be read as "for all universe levels, ...".

We write $h_n(X)$ to express that X is an n-type, or has homotopy level $(n + 2)$. Thus, n is an element of $\{-2, -1\} \cup \mathbb{N}$, and $h_n : U \to U$, or, more precisely,

$$h_n : \forall i,\ U_i \to U_i$$

is defined in the usual way:

$$h_0(X) :\equiv \text{is-contractible}(X)$$
$$h_{n+1}(X) :\equiv \forall x\, x' : X,\ h_n(x =_X x')$$

If x is a term of type X and n is a natural number, we want to write refl_x^n for iterating refl n-times on x. In MLTT, we cannot define that directly, but we have to define its type at the same time. We therefore use the usual definition of the *loop space*[2] $\Omega(X, x) :\equiv (x = x, \text{refl}_x)$, and the *iterated loop spaces* (where X is omitted in the

[1] http://red.cs.nott.ac.uk/~ngk/hierarchy/Hierarchy.html

[2] The HoTT book is arguable the best reference one can give here.

notation)

$$\Omega^0_x :\equiv (X, x)$$
$$\Omega^{n+1}_x :\equiv \Omega(\Omega^n_x)$$

Note that the type X can be some universe, and the term x can be some type. This will actually often be the case.

There are a couple of crucial lemmas about Ω which are omitted in this prose note. Instead of $\pi_1(\Omega^{n+1}_x)$, we write $\text{refl}^n_x = \text{refl}^n_x$, and instead of $\pi_2(\Omega^n_x)$, we write refl^n_x. [At least, in this early version of the note; at the moment, my partial Agda formalization also uses a slightly different definition of Ω, which I will probably change.]

Important Remark: Other Approaches and Previous Results

For $\neg h_1(U_1)$, a couple of people have found different solutions. I know of the following, but there might very well be others, to which I apologize:

- Peter L/Eric, with the following idea: To prove $\neg h_0(U_0)$, one can use the finite type 2; and for level $n + 1$, one can construct a type that contains only the type of level n, i.e. $T_{n+1} :\equiv \Sigma X : U_n. \|X = T_n\|$. This approach was also proposed by Vladimir in a discussion after a seminar talk (if I got it correct). It is not clear whether it leads to a solution. Eric and Peter have maybe solved more cases than $n \equiv 1$, I don't know.
- Thierry, using the type of $\mathbb{Z}/(2)$-sets (I hope I remember this correctly).
- Christian Sattler, with an approach by which the proof in this note is inspired, or even obtained as a generalization. We therefore sketch Christian's proof here:

  Define $A :\equiv \Sigma X : U_0, X = X$. Then, $h_1(U_1)$ implies $h_0(A = A)$, which implies $h_0(A \simeq A)$, which implies $h_{-1}(\text{id}_A = \text{id}_A)$, and therefore $h_{-1}(\forall (X, p) : A, (X, p) = (X, p)$. It now is sufficient to observe that $\lambda(X, p).\ \text{refl}_{(X,p)}$ is not the only function of this type: There is also $\lambda(X, p).\,(p, c)$, where c is a proof of $p * p = p$, obtained by a standard construction[3]. That these two functions are not equal follows from applying ("probing") them on $(2, \neg_2)$, the type with two inhabitants and the proof generated from the nontrivial automorphism with univalence.

Trying to use Christian's approach for cases higher than $n \equiv 1$ bares various difficulties. Even improving it for showing $\neg(h_2(U_2))$ is not at all simple. The arguable most severe obstacle is the necessity of the "transport"-argument in the end, which becomes horribly complicated, even for $n \equiv 2$. For the proof in this note, it is therefore quite important that this argument about transport can be avoided completely.

The Proof

Lemma 1. *Given a type B. Then, the types*

$$\text{refl}^{n+1}_B = \text{refl}^{n+1}_B$$

and

$$\forall (b : B),\ \text{refl}^n_b = \text{refl}^n_b$$

are isomorphic.

[3] by a standard lemma, $q * p$ can be written as composition of paths

Proof. We do induction on n. To make the induction go through, we also prove simultaneously that the constructed isomorphism maps

$$\mathrm{refl}_B^{n+2}$$

to

$$\lambda b.\ \mathrm{refl}_b^{n+1},$$

up to propositional equality.

For $n \equiv 0$, we have

$$\begin{aligned}
& \mathrm{refl}_B =_{B=B} \mathrm{refl}_B \\
& \text{(use the equivalence from } B = B \text{ to } B \approx B;\ c \text{ canonical proof of is-eq}(id)) \\
\approx\quad & (\mathrm{id}_B, c) =_{B\approx B} (\mathrm{id}_B, c) \\
\approx\quad & \Sigma(u : \mathrm{id}_B =_{B\to B} \mathrm{id}_B).\ u * c =_{\textit{is-weq}(\mathrm{id}_B)} c \\
& \text{(use that } \textit{is-weq}(\mathrm{id}_B) \text{ is a proposition)} \\
\approx\quad & \mathrm{id}_B =_{B\to B} \mathrm{id}_B \\
& \text{(use strong extensionality)} \\
\approx\quad & \forall b,\ b =_B b
\end{aligned}$$

By analyzing the steps, it is routine to check that refl_B^2 is mapped to $\lambda b.\ \mathrm{refl}_b$. This allows us to do the induction step, using the fact that an equivalence e preserves path spaces, i.e. $a = a'$ and $e(a) = e(a')$ are equivalent, with $a :\equiv a' :\equiv \mathrm{refl}_x^{n+2}$. □

Lemma 2. *If $n \geq 1$, then*

$$h_n(X) \to \forall(x : X),\ h_0(\mathrm{refl}_x^{n-1} = \mathrm{refl}_x^{n-1})$$

Proof. Induction on n.

For $n \equiv 1$, we need to shown

$$h_1(X) \to \forall x,\ h_0(x = x).$$

This is true by definition of the h-levels.

For the induction step, consider the statement for $n + 1$. It then becomes:

$$h_{n+1}(X) \to \forall(x : X),\ h_0(\mathrm{refl}_x^n = \mathrm{refl}_x^n).$$

Assume $h_{n+1}(X)$ and fix some $x : X$. We then have $h_n(x \equiv x)$, and, by the induction hypothesis,

$$\forall(p : x = x),\ h_0(\mathrm{refl}_p^{n-1} = \mathrm{refl}_p^{n-1}).$$

Use the special case $p :\equiv \mathrm{refl}_x$ to get the required result.

□

Corollary 3. *The following holds:*

$$\forall n : \mathbb{N},\ h_n(X) \to h_0(\mathrm{refl}_X^n = \mathrm{refl}_X^n)$$

and, as a consequence,

$$\forall n : \mathbb{N},\ h_0\left(h_n(X) \times (\mathrm{refl}_X^n = \mathrm{refl}_X^n)\right).$$

Proof. A corollary should not require an explicit proof, but here, I give one anyway for the first property. The case $n \equiv 0$ is simple. Fix $n \geq 1$ and assume $h_n(X)$. By Lemma 2,

$$\forall (x : X),\ h_0(\mathrm{refl}_x^{n-1} = \mathrm{refl}_x^{n-1}).$$

As is well-known, this implies

$$h_0(\forall (x : X),\ \mathrm{refl}_x^{n-1} = \mathrm{refl}_x^{n-1}),$$

and thus, by Lemma 1,

$$h_0(\mathrm{refl}_X^n = \mathrm{refl}_X^n).$$

□

Lemma 4. *For all X, $Y : X \to U_j$, $n \in \mathbb{N}$, the assumption $\forall (x : X), h_{n-1}(Yx)$ implies that, for all $(x, y) : \Sigma(x : X).\, Y(x)$,*

$$(\mathrm{refl}_{(x,y)}^n = \mathrm{refl}_{(x,y)}^n) \simeq (\mathrm{refl}_x^n = \mathrm{refl}_x^n)$$

Proof. Easy by induction on n, very similar to the proof of Lemma 1 - there could be a simplification (maybe by doing both at the same time). We just use that $\mathrm{refl}_{(x,y)}^{n+1}$ is an element of $\mathrm{refl}_{(x,y)}^n = \mathrm{refl}_{(x,y)}^n$, for which we already know the statement. □

Corollary 5. *Suppose Y is a family of sets over X.*[4] *For all $n > 0$ and for all $(x, y) : \Sigma XY$,*

$$(\mathrm{refl}_{(x,y)}^n = \mathrm{refl}_{(x,y)}^n) \simeq (\mathrm{refl}_x^n = \mathrm{refl}_x^n).$$

Lemma 6 (main lemma)**.** *For all $n : \mathbb{N}$, we define*

$$A_n :\equiv \Sigma(X : U_n)\,.\, h_n(X) \times (\mathrm{refl}_X^n = \mathrm{refl}_X^n).$$

We can construct terms of the following types:

- $b_n : h_{n+1}(A_n)$
- $r_n, s_n : \mathrm{refl}_{A_n}^{n+1} = \mathrm{refl}_{A_n}^{n+1}$
- $t_n : \neg(r_n = s_n)$

This can be done uniformly, so that we can in fact define a term of type

$$\forall n, (h_{n+1}(A_n)) \times \Sigma(r_n, s_n : \mathrm{refl}_{A_n}^{n+1} = \mathrm{refl}_{A_n}^{n+1}).\, \neg(r_n = s_n).$$

Proof. For b_n, use Lemma 6.1.1 from the book (the type of n-types is an $n + 1$-type) for the component X, and the lemmas above for the other component(s).

We just explain the induction step, as the case $n = 0$ corresponds to the proof of Christian Sattler, given above.

For r_n and s_n, we have to construct two terms of type $\mathrm{refl}_{A_n}^{n+1} = \mathrm{refl}_{A_n}^{n+1}$. By Lemma 1, this is equivalent to providing something of type $\forall ((X, b, p) : A_n),\ \mathrm{refl}_{(X,b,p)}^n = \mathrm{refl}_{(X,b,p)}^n$. Corollary 3 and Lemma 4 prove that type equivalent to

$$\forall ((X, b, p) : A),\ \mathrm{refl}_X^n = \mathrm{refl}_X^n\,.$$

[4] That is, $Y : X \to U$, and $\forall x,\ h_0(Y(x))$.

This type is inhabited by $\lambda(X, b, p).\, p$ and by $\lambda(X, b, p).\, \text{refl}_X^{n+1}$. We define s_n as the first, and r_n as the second. We have to prove that they are not equal. For that, observe that $(A_{n-1}, b_{n-1}, r_{n-1})$ and $(A_{n-1}, b_{n-1}, s_{n-1})$ are inhabitants of A_n. Assume $r_n = s_n$ and apply both on each of the two named inhabitants of A_n. This shows $r_{n-1} = \text{refl}_{A_{n-1}}^{n+1}$ and $s_{n-1} = \text{refl}_{A_{n-1}}^{n+1}$, implying $r_{n-1} = s_{n-1}$, thus contradicting t_{n-1}.

Remark: For convenience, we do not use that the proof obtained from s_n is the trivial one. If we did, we would have to show again that the canonical isomorphism preserves trivial inhabitants. □

Theorem 7 (Solution to question number 2 on the UF list of open problems)**.** *A_n is an $n+1$-type, but not an n-type; i.e. the type*

$$\forall n,\ \Sigma(T : U_{n+1}).\, h_{n+1}(T) \times \neg(h_n(T))$$

is inhabited.

Proof of Theorem 7. For a given n, choose $T :\equiv A_n$. Then, $h_{n+1}(A_n)$ is shown by the term b_n. Assume $h_n(A_n)$. By Corollary 3, we have $h_0(\text{refl}_{A_n}^n = \text{refl}_{A_n}^n)$. But this would imply $h_{-1}(\text{refl}_{A_n}^{n+1} = \text{refl}_{A_n}^{n+1})$, which is contradicted by r_n, s_n and t_n. □

Theorem 8.

$$\forall n,\ \neg h_n(U_n)$$

Proof of Theorem 8. Assume $h_{n+1}(U_{n+1})$. By Lemma 2, this implies $h_0(\text{refl}_{A_n}^n = \text{refl}_{A_n}^n)$, allowing us to use the same argument as above. □